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Proximinality in $L_p(\mu, X)$

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Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leqslant p \leqslant \infty$ and let us denote by $L_p(\mu, X)$ the Banach space of all X-valued Bochner p-integrable (essentially bounded for $p = \infty$) functions on a certain positive complete σ-finite measure space (Ω, Σ, μ), endowed with the usual p-norm. In this paper we give a negative answer to the following question: "If Y is proximinal in X, is $L_p(\mu, Y)$ proximinal in $L_p(\mu, X)$?" We also show that the answer is affirmative for separable spaces Y. Some consequences of this are obtained.

1. INTRODUCTION

Let X be a Banach space and let Y be a closed subspace of X. Let us recall that we say that Y is proximinal in X if for each $x \in X$ there exists $y \in Y$ such that

$$\|x - y\| = \operatorname{dist}(x, Y) = \inf\{\|x - z\| : z \in Y\}.$$

In this case y is called a best approximation of x in Y. If this best approximation is unique for all $x \in X$, then Y is said to be Chebyshev.

In this paper (Ω, Σ, μ) stands for a complete positive σ-finite measure space, and we assume $1 \leqslant p \leqslant \infty$. We denote by $L_p(\mu, X)$ the Banach space of all Bochner p-integrable (essentially bounded for $p = \infty$) functions on Ω with values in X, endowed with the usual p-norm. We simply denote $L_p(\mu)$, when X is the scalar field.

Several papers have been devoted to studying when the space $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$ (see the references), and, in the words of [7],

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"the main problem that these papers address is: If Y is proximinal in X, is $L_p(\mu, Y)$ proximinal in $L_p(\mu, X)$?" Only partial answers to this question have been given. In this paper we solve this problem providing an example which shows that the answer is negative. It uses a Banach space already considered by Holmes and Kripke [3, Example 4].

Once we know this example, it seems interesting to know for which Banach spaces the answer is yes. In fact several answers have already been given in the literature (see the references). In this direction we show that (Theorem 3.4)

If Y is separable then $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$ if (and only if) Y is proximinal in X $\quad (*)$

We obtain some consequences from this. We have to mention that this result as well as a pointwise version of it (Theorem 3.2) have already been stated in [12] (Theorems 3.4 and 3.5 of [12]); however, the proof given there has a gap (see Remark 2.5). We follow a completely different approach, using a technique we learned from Hu and Lin (see [4]).

We would like to point out that $(*)$ generalizes and improves several previously known results on the subject (see the references).

Our notation is standard, as in [2]. In particular $\|\cdot\|_p$ is the usual p-norm in $L_p(\mu, X)$ or in $L_p(\mu)$.

2. PRELIMINARIES

In this section we wish to include some facts which are fundamental in our study. We begin with three results about distances and best approximations in $L_p(\mu, X)$ which were proved in [8].

Theorem 2.1 (Theorem 5 of [8]). *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leqslant p \leqslant \infty$ and let $f \in L_p(\mu, X)$. Then the function $s \mapsto \operatorname{dist}(f(s), Y)$, which we denote $\operatorname{dist}(f(\cdot), Y)$, is measurable, and*

$$\operatorname{dist}(f, L_p(\mu, Y)) = \|\operatorname{dist}(f(\cdot), Y)\|_p.$$

Corollary 2.2 (Corollary 2 of [8]). *Let X be a Banach space, Y a closed subspace of X, and $1 \leqslant p < +\infty$. Let $f \in L_p(\mu, X)$ and let $g \in L_p(\mu, Y)$. Then g is a best approximation of f in $L_p(\mu, Y)$ if and only if $g(s)$ is a best approximation of $f(s)$ in Y for almost all s.*

LEMMA 2.3 (Lemma 1 of [8]). *Let X be a Banach space, Y a closed subspace of X, and $1 \leqslant p \leqslant +\infty$. Let $f \in L_p(\mu, X)$ and let $g: \Omega \to Y$ be a measurable function such that $g(s)$ is a best approximation of $f(s)$ in Y for almost all s. Then g is a best approximation of f in $L_p(\mu, Y)$ (and therefore $g \in L_p(\mu, Y)$).*

Although it is not important for our purposes, we wish to point out that in the statement of the preceding result in [8] one reads "proximinal" instead of "closed." A look at the proof shows that it is indeed enough to state "closed."

For $p = +\infty$ the condition in Corollary 2.2 is sufficient (see the preceding lemma), but one realises easily that it is necessary only in very trivial situations (see Remark 3.10). To overcome this difficulty we will prove a result (Proposition 2.5) which will play a role similar to Corollary 2.2, but first we need the following easy lemma.

LEMMA 2.4. *Let X be a Banach space, Y a closed subspace of X, $A \in \Sigma$ a set with positive measure, and let $f \in L_\infty(\mu, X)$ be such that*

$$\operatorname{dist}(f(s), Y) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \in \Omega \setminus A. \end{cases}$$

Then

$$\operatorname{dist}(f, L_\infty(\mu, Y)) = 1.$$

Proof. It is clear that

$$\operatorname{dist}(f, L_\infty(\mu, Y)) \geqslant 1.$$

For the converse inequality, given $\varepsilon > 0$ take a measurable countably valued function $f_0 \in L_\infty(\mu, X)$ such that $f_0(\Omega) \subset f(\Omega)$ and

$$\|f - f_0\|_\infty < \frac{\varepsilon}{2}.$$

The function f_0 has the form

$$f_0 = \sum_{n=1}^{\infty} \chi_{B_n}(\cdot)\, x_n,$$

with the B_n's disjoint and measurable, and $x_n \in f(\Omega)$. For each n take $y_n \in Y$ such that

$$\|x_n - y_n\| < \operatorname{dist}(x_n, Y) + \frac{\varepsilon}{2} \leqslant 1 + \frac{\varepsilon}{2}.$$

It is clear that g defined by

$$g = \sum_{n=1}^{\infty} \chi_{B_n}(\cdot)\, y_n$$

belongs to $L_\infty(\mu, Y)$ and satisfies

$$\|f - g\|_\infty \leqslant \|f - f_0\|_\infty + \|f_0 - g\|_\infty \leqslant 1 + \varepsilon.$$

Therefore,

$$\operatorname{dist}(f, L_\infty(\mu, Y)) \leqslant 1 + \varepsilon.$$

PROPOSITION 2.5. *Let X be a Banach space and let Y be a closed subspace of X. Then $L_\infty(\mu, Y)$ is proximinal in $L_\infty(\mu, X)$ if and only if for each $f \in L_\infty(\mu, X)$ there exists $g \in L_\infty(\mu, Y)$ such that $g(s)$ is a best approximation of $f(s)$ in Y for almost all s.*

Proof. Sufficiency of the condition is an immediate consequence of Lemma 2.3. Let us show its necessity. Assume that $L_\infty(\mu, Y)$ is proximinal in $L_\infty(\mu, X)$ and take $f \in L_\infty(\mu, X)$. Consider the non-negative measurable function

$$h\colon \Omega \to [0, +\infty)$$
$$s \to \operatorname{dist}(f(s), Y).$$

Take $\Omega_0 = \{s \in \Omega\colon h(s) = 0\}$ and for each $n = 1, 2, ...,$ take $\Omega_n = \{s \in \Omega\colon n - 1 < h(s) \leqslant n\}$. Of course, we may forget those Ω_n which are μ-null sets, so, without loss of generality, we will assume $\mu(\Omega_n) > 0$ for all n. Now, for each $n = 1, 2, ...$ we define $f_n\colon \Omega \to X$ by

$$f_n(s) = \begin{cases} \dfrac{1}{h(s)} f(s) & \text{if } \ s \in \Omega_n \\ 0 & \text{if } \ s \in \Omega \backslash \Omega_n. \end{cases}$$

It is clear that f_n belongs to $L_\infty(\mu, X)$ and also that

$$\operatorname{dist}(f_n(s), Y) = \operatorname{dist}\left(\frac{1}{h(s)} f(s), Y\right) = \frac{1}{h(s)} \operatorname{dist}(f(s), Y) = 1$$

for all $s \in \Omega_n$. So, it follows from the preceding lemma that

$$\operatorname{dist}(f_n, L_\infty(\mu, Y)) = 1.$$

On the other hand, using the proximinality of $L_\infty(\mu, Y)$, we deduce that there exists $g_n \in L_\infty(\mu, Y)$ such that

$$\|f_n - g_n\|_\infty = \operatorname{dist}(f_n, L_\infty(\mu, Y)) = 1.$$

Therefore, we have

$$1 = \operatorname{dist}(f_n(s), Y) \leqslant \|f_n(s) - g_n(s)\| \leqslant \|f_n - g_n\|_\infty = 1$$

for almost all $s \in \Omega_n$, and so

$$\|f_n(s) - g_n(s)\| = 1$$

for almost all $s \in \Omega_n$. This implies that

$$\operatorname{dist}(f(s), Y) = h(s) = h(s)\ \|f_n(s) - g_n(s)\| = \|f(s) - h(s)\ g_n(s)\|$$

for almost all $s \in \Omega_n$. Now it is clear that g defined by

$$g(s) = \chi_{\Omega_0}(s)\ f(s) + \sum_{n=1}^{\infty} \chi_{\Omega_n}(s)\ h(s)\ g_n(s)$$

for all $s \in \Omega$ enjoys the required property.

The next result is very well known (see for instance [12, Theorem 3.3] or [1, p. 231]). We would like to point out that it is an immediate consequence of Corollary 2.2 (in the case $1 \leqslant p < +\infty$) and the preceding proposition (in the case $p = +\infty$), just taking constant functions.

Corollary 2.6. *Let X be a Banach space, Y a closed subspace of X, and $1 \leqslant p \leqslant +\infty$. Let us suppose that $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$. Then Y is proximinal in X.*

The following corollary is essentially known (see [6, Theorem 1.1; 1, Theorem 1.1]), but we think that the implication (i) $\Rightarrow$ (ii) in the case $p = +\infty$ is new. This is why we have included the proof.

Corollary 2.7. *Let X be a Banach space and let Y be a closed subspace of X. Let $1 \leqslant p \leqslant \infty$. The following are equivalent*:

(i) *$L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$*

(ii) *$L_1(\mu, Y)$ is proximinal in $L_1(\mu, X)$.*

Proof. Assume $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$. Take $f \in L_1(\mu, X)$ and let us show that there is a best approximation of f in $L_1(\mu, Y)$. Let

$\{A_m\}$ be a countable partition of Ω in measurable sets of finite measure. For each natural number n let $A_{mn} = \{s \in A_m : n-1 \leqslant \|f(s)\| < n\}$, and let $f_{mn} = \chi_{A_{mn}} f$. Of course f_{mn} belongs to $L_p(\mu, X)$ and then it has a best approximation g_{mn} in $L_p(\mu, Y)$. In the case $1 \leqslant p < +\infty$ this means that $g_{mn}(s)$ is a best approximation of $f_{mn}(s)$ in Y for almost all s (see Corollary 2.2), and in the case $p = +\infty$, Proposition 2.5 says that we may assume $g_{mn}(s)$ has the same property. Define now $g = \sum_{mn} g_{mn}$. It is clear that g is measurable and $g(s)$ is a best approximation of $f(s) = \sum_{mn} f_{mn}(s)$ in Y for almost all s. It follows from Lemma 2.3 that g is indeed a best approximation of f in $L_1(\mu, Y)$.

Observe that the converse may be proved in an analogous way.

Remark 2.8. Let us explain now where the gap is in [12]. The proof of Theorems 3.4 and 3.5 of [12] (our Theorems 3.2 and 3.4) given by You and Guo lies in their Lemma 2.1. However, this lemma is a misquotation of Theorem 1′ of their Ref. [6], and it is false as is stated. The problem is the following: in Theorem 1′ of Ref. [6] of You and Guo "measurable" actually means "compactly measurable" (a function is compactly measurable if inverse images of compact sets are measurable sets), and therefore the V_n's in Lemma 2.1 of [12] should be compactly measurable functions instead of measurable functions in the usual sense (functions whose inverse images of open sets are measurable sets). Notice that if we apply Lemma 2.1 of [12] to (single valued) functions, it says that every compactly measurable function is measurable, but this is not true. Let us give a simple example. Let X be any infinite dimensional Banach space, and let us consider the following measure space: Ω is B_X, the unit ball of X, Σ is the class of all subsets of B_X which are Baire first category or has a Baire first category complement, and μ is the measure which takes the value 1 in B_X and 0 in all Baire first category subsets of B_X. Take $f: \Omega \to X$ the identity function. Of course it is compactly measurable, but it is not measurable because $f^{-1}(\frac{1}{2}B_X) = \frac{1}{2}B_X$ is not measurable. Actually, in a not so easy way we could also consider examples in the Lebesgue space of measure $[0, 1]$.

Now we will give a lemma which is the basis of the technique developed by Hu and Lin [4]. We will use it in the proof of Theorem 3.2. We have included its proof just for the sake of completeness. Of course diam B denotes the diameter of the set $B \subset X$, that is,

$$\operatorname{diam} B = \operatorname{Sup}\{\|x - y\| : x, y \in B\}.$$

LEMMA 2.9 (Lemma 3 of [4]). *Assume $\mu(\Omega) < +\infty$. Suppose (M, d) is a metric space and A a subset of Ω such that $\mu^*(A) = \mu(\Omega)$, where μ^* denotes the outer measure associated to μ. If g is a mapping from Ω to*

M with separable range, then for any $\varepsilon > 0$ there exist a countable partition $\{E_n\}$ of Ω in measurable sets and $A_n \subset A \cap E_n$ such that $\mu^*(A_n) = \mu(E_n)$ and diam $g(A_n) < \varepsilon$ *for all n.*

Proof. Let $\varepsilon > 0$ and let us assume $\mu(\Omega) > 0$ (otherwise it is trivial). For a subset D of A, we define $D(\varepsilon)$ as the class of all subsets G of D such that diam $g(G) < \varepsilon$, and

$$M(D) = \operatorname{Sup}\{\mu^*(G) \colon G \in D(\varepsilon)\}.$$

Since $g(A)$ is separable, we have $M(D) > 0$ whenever $\mu^*(D) > 0$. Choose $A_1 \in A(\varepsilon)$ such that $\mu^*(A_1) \geqslant \frac{1}{2} M(A)$. There is a measurable cover B_1 of A_1. By induction, there exist a sequence $\{A_n\}$ of subsets of A and a sequence of pairwise disjoint measurable sets $\{B_n\}$ such that

$$A_n \in \left(A \setminus \bigcup_{i<n} B_i\right)(\varepsilon), \qquad A_n \subset B_n, \qquad \text{and}$$

$$\mu^*(A_n) = \mu(B_n) \geqslant \tfrac{1}{2} M\left(A \setminus \bigcup_{i<n} B_i\right)$$

for all n. Since $\mu(\Omega) < +\infty$, we have $\lim_n \mu(B_n) = 0$, and so $\lim_n \mu^*(A_n) = 0$, which implies that $\lim_n M(A \setminus \bigcup_{i<n} B_i) = 0$. Hence $M(A \setminus \bigcup_n B_n = 0)$, and so $\mu^*(A \setminus \bigcup_n B_n) = 0$. But we have

$$\begin{aligned}
\mu(\Omega) &= \mu\left(\bigcup_n B_n\right) + \mu\left(\Omega \setminus \bigcup_n B_n\right) = \mu^*(A) \\
&\leqslant \mu^*\left(A \setminus \bigcup_n B_n\right) + \mu^*\left(A \cap \left(\bigcup_n B_n\right)\right) \\
&= \mu^*\left(A \cap \left(\bigcup_n B_n\right)\right) \leqslant \mu\left(\bigcup_n B_n\right).
\end{aligned}$$

Therefore $\mu(\Omega \setminus \bigcup_n B_n) = 0$. Let $E_1 = B_1 \cup (\Omega \setminus \bigcup_n B_n)$ and $E_n = B_n$ for $n > 1$. Then $\{E_n\}$ is a partition of Ω, $A_n \subset A \cap E_n$ with diam $g(A_n) < \varepsilon$, and $\mu^*(A_n) = \mu(E_n)$ for all n. This completes the proof.

3. MAIN RESULTS

Let us begin with an example showing that Y proximinal in X does not imply $L_p([0, 1], Y)$ proximinal in $L_p([0, 1], X)$. We use a Banach space X_0 and its subspace Y_0 already considered by Holmes and Kripke [3].

EXAMPLE 3.1. In [3, Example 4] it is shown that in each $l_p(3)$ (the three dimensional l_p) with $1 < p < \infty$, we can find vectors m_p, x_p, and y_p such that

(i) $\operatorname{Sup}\{\|m_p\|_p, \|x_p\|_p, \|y_p\|_p : 1 < p < \infty\} < +\infty$, where of course $\|\cdot\|_p$ denotes here the norm in $l_p(3)$.

(ii) If F_p is the line in $l_p(3)$ spanned by m_p and P_{F_p} is the *best approximation operator* (or *metric projection operator*) *supported by* F_p (that is, for each $x \in l_p(3)$, $P_{F_p}(x)$ is the unique best approximation of x in F_p), then $P_{F_p}(x_p) = 0$ and $\|P_{F_p}(x_p + y_p)\|_p \geqslant 1$ for all p.

(iii) $\|y_p\|_p$ decreases to 0 as p increases.

Let us denote by X_0 the Banach space of all sequences $\{z_p\}_{p \geqslant 2}$ such that $z_p \in l_p(3)$ for all natural numbers $p \geqslant 2$ and $\{\|z_p\|_p\}$ is bounded, endowed with the norm

$$\|\{z_p\}\| = \operatorname{Sup}\{\|z_p\|_p : p = 2, 3, \ldots\} + \left(\sum_{p=2}^{\infty} \frac{\|z_p\|_p^2}{p^2}\right)^{1/2}.$$

The space X_0 is strictly convex. Let Y_0 be the subspace of X_0 of all sequences of the form $\{\lambda_p m_p\}$ with $\{\lambda_p\} \in l_\infty$. It is very easy to show that Y_0 is a Chebyshev subspace of X_0 and that P_{Y_0}, the best approximation operator supported by Y_0, is defined by

$$P_{Y_0}(\{z_p\}) = \{P_{F_p}(z_p)\} \qquad \text{for all} \quad \{z_p\} \in X_0.$$

Let us define $f\colon [0, 1] \to X_0$ by

$$f(t) = \{x_p + a_p(t)\, y_p\}_{p \geqslant 2},$$

where $\{a_p(t)\}_{p \geqslant 2}$ is the sequence of digits in the binary expression of t, that is,

$$a_2(t) = \chi_{\{1\}}(t), \qquad a_3(t) = \chi_{[1/2,\, 1)}(t),$$
$$a_4(t) = \chi_{[1/4,\, 1/2) \cup [3/4,\, 1)}(t), \ldots, a_p(t) = \chi_{A_p}(t), \ldots,$$

where $A_p = [1/2^{p-2}, 2/2^{p-2}) \cup [3/2^{p-2}, 4/2^{p-2}) \cup \cdots \cup [(2^{p-2} - 1)/2^{p-2}, 2^{p-2}/2^{p-2})$. Since $\lim_p \|y_p\|_p = 0$, f is the uniform limit of the functions f_n of the form

$$f_n(t) = \{x_p\}_{p \geqslant 2} + \{\chi_{A_2}(t)\, y_2, \chi_{A_3}(t)\, y_3, \ldots, \chi_{A_n}(t)\, y_n, 0, 0, \ldots\}.$$

It is clear that f is measurable and bounded and therefore, it belongs to all $L_p([0, 1], X_0)$ for $1 \leqslant p \leqslant \infty$. Let us suppose that f has a best approximation g in $L_p([0, 1], Y_0)$. By Corollary 2.2 and Proposition 2.5 we may assume that g and $(P_{Y_0} \circ f)$ coincide almost everywhere, and therefore $(P_{Y_0} \circ f)$ should be measurable and hence essentially separably valued. Let us see that this is not the case. Let Z be a null subset of $[0, 1]$. It is clear that $[0, 1] \backslash Z$ must contain a non-denumerable set $\{t_\alpha\}$. Notice that for $\alpha \neq \beta$ there exists $p_0 \geqslant 2$ such that

$$a_{p_0}(t_\alpha) = \chi_{A_{p_0}}(t_\alpha) \neq a_{p_0}(t_\beta) = \chi_{A_{p_0}}(t_\beta)$$

therefore,

$$\begin{aligned}
&\|(P_{Y_0} \circ f)(t_\alpha) - (P_{Y_0} \circ f)(t_\beta)\| \\
&\quad = \|P_{Y_0}(\{x_p\}_{p \geqslant 2} + \{a_2(t_\alpha)\, y_2, a_3(t_\alpha)\, y_3, ..., a_n(t_\alpha)\, y_n, ...\}) \\
&\qquad - P_{Y_0}(\{x_p\}_{p \geqslant 2} + \{a_2(t_\beta)\, y_2, a_3(t_\beta)\, y_3, ..., a_n(t_\beta)\, y_n, ...\})\| \\
&\quad = \|\{P_{F_2}(x_2 + a_2(t_\alpha)\, y_2), P_{F_3}(x_3 + a_3(t_\alpha)\, y_3), ..., \\
&\qquad P_{F_n}(x_n + a_n(t_\alpha)\, y_n), ...\} \\
&\qquad - \{P_{F_2}(x_2 + a_2(t_\beta)\, y_2), P_{F_3}(x_3 + a_3(t_\beta)\, y_3), ..., \\
&\qquad P_{F_n}(x_n + a_n(t_\beta)\, y_n), ...\}\| \\
&\quad \geqslant \|P_{F_{p_0}}(x_{p_0} + a_{p_0}(t_\alpha)\, y_{p_0}) - P_{F_{p_0}}(x_{p_0} + a_{p_0}(t_\beta)\, y_{p_0})\|_{p_0} \\
&\quad = \|P_{F_{p_0}}(x_{p_0} + y_{p_0})\|_{p_0} \geqslant 1
\end{aligned}$$

and of course this means that $(P_{Y_0} \circ f)([0, 1] \backslash Z)$ is not separable. Therefore $(P_{Y_0} \circ f)$ is not essentially separably valued and cannot be measurable.

Let us give now our positive results on proximinality in $L_p(\mu, X)$-spaces. The first is a pointwise version. The $\|\cdot\|_p$-version will follow from it.

THEOREM 3.2. *Let X be a Banach space and let Y be a closed separable subspace of X. Let us suppose that Y is proximinal in X and let $f\colon \Omega \to X$ be a measurable function. Then there is a measurable function $g\colon \Omega \to Y$ such that $g(s)$ is a best approximation of $f(s)$ in Y for almost all s.*

Proof. Since f is measurable, we may assume it is separably valued, therefore, using also that μ is σ-finite, we can find a countable partition $\{\Omega_n\}$ of Ω in measurable sets in such a way that

$$\mu(\Omega_n) < +\infty \qquad \text{and} \qquad \operatorname{diam} f(\Omega_n) < \tfrac{1}{2} \qquad \text{for all } n.$$

For each $s \in \Omega$ let $g_0(s) \in Y$ be a best approximation of $f(s)$ in Y. Let us apply Lemma 2.9 to the mapping $g_0 : \Omega \to Y$ in each Ω_n, taking $\varepsilon = \frac{1}{2}$ and $\Omega = A = \Omega_n$. We get countable partitions in each Ω_n, and therefore a countable partition in the whole Ω. That is, we get a countable partition $\{E_n\}$ of Ω in measurable sets, and a sequence $\{A_n\}$ of subsets of Ω, such that

$$A_n \subset E_n, \qquad \mu^*(A_n) = \mu(E_n) < +\infty,$$

$$\operatorname{diam} g_0(A_n) < \tfrac{1}{2}, \qquad \text{and} \qquad \operatorname{diam} f(E_n) < \tfrac{1}{2}.$$

Let us apply again the same argument in each E_n, with $\varepsilon = 1/2^2$, $\Omega = E_n$, and $A = A_n$. For each n we get a countable partition $\{E_{nk}\}$ of E_n in measurable sets, and a sequence $\{A_{nk}\}$ of subsets of Ω such that

$$A_{nk} \subset A_n \cap E_{nk}, \qquad \mu^*(A_{nk}) = \mu(E_{nk})$$

$$\operatorname{diam} g_0(A_{nk}) < \frac{1}{2^2} \qquad \text{and} \qquad \operatorname{diam} f(E_{nk}) < \frac{1}{2^2}$$

for all n and k. Let us proceed by induction. For each natural number k let Δ_k be the set of all k-tuples of natural numbers, and let $\Delta = \bigcup_k \Delta_k$. Let us consider in Δ the partial order defined by

$$(i_1, ..., i_m) \leqslant (j_1, ..., j_n) \quad \text{if and only if } m \leqslant n \text{ and } i_k = j_k \text{ for } k = 1, ..., m.$$

Then, by induction, for each natural number k we can take a partition $\{E_\alpha\}_{\alpha \in \Delta_k}$ of Ω in measurable sets and a collection $\{A_\alpha\}_{\alpha \in \Delta_k}$ of subsets of Ω in such a way that

(1) $A_\alpha \subset E_\alpha$ and $\mu^*(A_\alpha) = \mu(E_\alpha)$ for each α

(2) $E_\beta \subset E_\alpha$ and $A_\beta \subset A_\alpha$ if $\alpha \leqslant \beta$

(3) $\operatorname{diam} f(E_\alpha) < 1/2^k$ and $\operatorname{diam} g_0(A_\alpha) < 1/2^k$ if $\alpha \in \Delta_k$.

For each $\alpha \in \Delta$ take $s_\alpha \in A_\alpha$ (forget the α's for which $A_\alpha = \phi$). For each k we define

$$g_k(\cdot) = \sum_{\alpha \in \Delta_k} \chi_{E_\alpha}(\cdot)\, g_0(s_\alpha).$$

Using (1), (2), and (3) it is easy to see that, for all $s \in \Omega$, $\{g_k(s)\}$ is a Cauchy sequence in Y, and therefore a convergent one. Let $g : \Omega \to Y$ be the pointwise limit of $\{g_k\}$. Of course g is measurable. Let $s \in \Omega$ and k be a natural number. Suppose $s \in E_\alpha$. We have

$$\begin{aligned}\|f(s)-g_k(s)\| &= \|f(s)-g_0(s_\alpha)\| \\ &\leqslant \|f(s)-f(s_\alpha)\| + \|f(s_\alpha)-g_0(s_\alpha)\| \\ &< \frac{1}{2^k} + \operatorname{dist}(f(s_\alpha), Y) \\ &\leqslant \frac{1}{2^k} + \operatorname{dist}(f(s), Y) + \|f(s)-f(s_\alpha)\| \\ &< \operatorname{dist}(f(s), Y) + \frac{1}{2^{k-1}}.\end{aligned}$$

Therefore

$$\|f(s)-g(s)\| = \lim_k \|f(s)-g_k(s)\| = \operatorname{dist}(f(s), Y)$$

and so $g(s)$ is a best approximation of $f(s)$ in Y. This completes the proof.

Remark 3.3. Example 3.1 shows that in the preceding theorem the separability assumption cannot be removed.

THEOREM 3.4. *Let X be a Banach space, let Y be a closed separable subspace of X, and let $1 \leqslant p \leqslant \infty$. Then $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$ if and only if Y is proximinal in X.*

Proof. Necessity is Corollary 2.6. Let us show sufficiency. Let us suppose that Y is proximinal in X, and let f be a function in $L_p(\mu, X)$. The preceding theorem guarantees that there exists a measurable function g defined on Ω with values in Y such that $g(s)$ is a best approximation of $f(s)$ in Y for almost all s. It follows from Lemma 2.3 that g is a best approximation of f in $L_p(\mu, Y)$.

COROLLARY 3.5. *Let X be a Banach space, let Y be closed subspace of X, and let $1 \leqslant p \leqslant \infty$. Let us suppose that each separable subspace of Y is proximinal in X. Then $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$.*

Proof. Let us suppose that each separable subspace of Y is proximinal in X. Let $f \in L_p(\mu, X)$. There exists a sequence $\{g_n\} \subset L_p(\mu, Y)$ such that $\operatorname{dist}(f, L_p(\mu, Y)) = \lim_n \|f - g_n\|_p$. We may assume that all g_n's are separably valued and so we will assume that there is a separable subspace Y_1 of Y such that $\{g_n\} \subset L_p(\mu, Y_1)$. By our hypothesis and Theorem 3.4, there exists a best approximation g of f in $L_p(\mu, Y_1)$. But then g is also a best approximation of f in $L_p(\mu, Y)$ because

$$\operatorname{dist}(f, L_p(\mu, Y)) = \lim_n \|f - g_n\|_p \geqslant \operatorname{dist}(f, L_p(\mu, Y_1))$$
$$\geqslant \operatorname{dist}(f, L_p(\mu, Y)).$$

This completes the proof.

Remark 3.6. In the preceding corollary the hypothesis "each separable subspace of Y is proximinal in X" may be substituted by the slightly more general condition "each separable subspace of Y is contained in a separable subspace of Y which is proximinal in X."

COROLLARY 3.7 (Theorem 1.2 of [5], Corollary 1.2 of [1]). *Let X be a Banach space, let $1 \leqslant p \leqslant \infty$, and let Y be a reflexive subspace of X. Then $L_p(\mu, Y)$ is proximinal in $L_p(\mu, X)$.*

Proof. We are obviously in the hypothesis of the preceding corollary.

Remark 3.8. Notice that in the preceding corollary the only non-trivial cases are $p = 1$ and $p = \infty$, because otherwise $L_p(\mu, Y)$ is reflexive.

COROLLARY 3.9. *Let X be a Banach space, let Y be a separable subspace of X, and let $1 \leqslant p < \infty$. Then $L_p(\mu, Y)$ is Chebyshev in $L_p(\mu, X)$ if and only if Y is Chebyshev in X.*

Proof. Existence of a best approximation follows from Theorem 3.4 and uniqueness from Corollary 2.2.

Remark 3.10. It should be pointed out that $L_\infty(\mu, Y)$ is Chebyshev in $L_\infty(\mu, X)$ only in completely trivial situations. Let us see why. Assume that there are two disjoint measurable sets, A_1, A_2, with positive measure (otherwise the measure space (Ω, Σ, μ) would be trivial), and assume that Y is a Chebyshev subspace of X such that $Y \neq \{0\}$, $Y \neq X$ (we are again excluding trivial cases). We will show that $L_\infty(\mu, Y)$ is not Chebyshev in $L_\infty(\mu, X)$. Take any $x \in X \setminus Y$ with a non-zero best approximation y in Y (it is immediate that there are infinite vectors x with this property), and take any $\varepsilon > 0$ such that

$$\varepsilon < \frac{\|x - y\|}{\|y\|}.$$

It is straightforward to show that if we denote

$$f = \chi_{A_1}(\cdot)\, x + \chi_{A_2}(\cdot)\, y,$$

then

$$g_1 = \chi_{A_1 \cup A_2}(\cdot)\, y \qquad \text{and} \qquad g_2 = \chi_{A_1}(\cdot)\, y + (1 + \varepsilon)\, \chi_{A_2}(\cdot)\, y$$

are two different best approximations of f in $L_\infty(\mu, Y)$. In other words $L_\infty(\mu, Y)$ is not Chebyshev in $L_\infty(\mu, X)$.

Final Remark. After reading the first version of this paper, D. H. Fremlin (personal communication) kindly showed the author that Theorem 3.2 could be also proved using measurable selection theorems.

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